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STUDY OF THE GEOMETRIC STRUCTURE OF PHASE PORTRAITS OF A NONLINEAR SYSTEM IN A PLANE

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Abstract *– the article is devoted to the study of the mathematical model given by the system of nonlinear differential equations for the purpose of identifying its equilibrium points and conducting an analysis of the stability of these points in accordance with the Lyapunov stability method. The performed studies allow to provide a geometric interpretation of the behavior of the system and to visually present its phase portraits. Such an analysis makes it possible to predict the modes of operation of the system under different initial conditions. The research of the mathematical model is carried out by numerical methods in the MATLAB system for engineering and mathematical calculations using the integration of the system of differential equations by the Runge-Kut method of the 4th order.*

Keywords*– nonlinear dynamics, phase portraits, stability according to Lyapunov.*

Problem Statement. The level of professional training of an engineer is largely determined by his ability to conduct research with modern tools, including using such powerful systems for engineering and mathematical calculations as MATLAB [1-4]. Within the discipline "Software for mathematical calculations", which is taught for the specialty "Mathematics" in the second year of the National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute", students learn to conduct research on given mathematical models represented by a system of differential equations, not only by using analytical methods and numerical integration, but also using a geometric interpretation of the behavior of the system in the phase space, which expands the general picture for predicting the behavior of such systems under different initial conditions.

Analysis of Recent Research. The study of mathematical models represented by systems of differential equations is carried out analytically, or, in cases where the systems of equations are not integrated, using numerical integration. Each numerical integration allows you to build a single phase trajectory, which does not give a complete picture of the system's behavior. Construction of phase portraits of the studied system as a set of characteristic trajectories, analysis of their structure, supplementing the conducted research with the determination of the stability conditions of the equilibrium points of the system is a powerful tool for understanding the general picture of the behavior of the system and predicting its behavior under different initial conditions [5-9].

Formulation of Goals. When studying the discipline "Software for mathematical calculations", students master various methods of analysis and research of mathematical models given by the system of differential equations. The purpose of the article is to provide an example of such a study conducted within the scope of the specified course. At the same time, numerical integration of the system of equations was used and the stability conditions of the constructed equilibrium points of the system were analyzed. The result of the conducted research was the construction of a phase portrait of the system, which made it possible to predict the behavior of the system under the given initial conditions.

Main Part. The complex of laboratory works developed in the course allows students to acquire the skills of conducting a multifaceted study of the problem using various methods to confirm the reliability of the obtained results by comparing them. One of the laboratory works of the course consists in the fact that for a given mathematical model represented by a system of differential equations, it is necessary to conduct the following complex research. Let's consider this on a concrete example.

In the version of the laboratory work, a system of differential equations was set::

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = x + x^2$$

In the process of studying the system, we search for an analytical solution, if it exists, and we study the given system using numerical methods.

Analytical solution of the system gave the following results:

x=C2((2*exp(4*t)*sin(11^(1/2)*t))/5-*
*(11^(1/2)*exp(4*t)*cos(11^(1/2)*t))/5)-...*
C1((2*exp(4*t)*cos(11^(1/2)*t))/5 +*
*(11^(1/2)*exp(4*t)*sin(11^(1/2)*t))/5)*
*y =C1*exp(4*t)*cos(11^(1/2)*t) - C2*exp(4*t)*sin(11^(1/2)*t)*

From the obtained analytical solution, a series of integral curves can be constructed to explain the behavior of the system, but the choice of initial conditions is not obvious. Therefore, it is desirable to carry out a qualitative assessment of the system in order to set the initial conditions based on the received data. This will make it possible to more fully reveal the behavior of the system.

We will conduct a study of the system for the presence of equilibrium points and examine them for stability using the Lyapunov stability theory. First, we will find the equilibrium points of the system. To do this, we are looking for a solution to the following system of equations obtained from the original one with zero value of the derivatives:

$$y = 0$$

$$x + x^2 = 0$$

As a result, we have two equilibrium points with *x* and *y* coordinates:

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The first point   0,0
The second point -1,0
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We will conduct a study of equilibrium points for stability. To do this, we linearize the given system and substitute the values of the coordinates of each of the obtained equilibrium points. According to the method of stability of equilibrium points by the Lyapunov method, we can draw the following conclusions. The first point of balance (x=0, y=0) is *saddle*, since the obtained values of the roots of the age equation for the linearization matrix at this point take values λ_1=-1, λ_2=1. This means that in the phase space there is a direction of attraction to the equilibrium point, which is indicated by the root of the age equation with the value λ_1=-1 and a direction of repulsion from the equilibrium point, which is indicated by the root of the age equation with a value of λ_2=+1. The second equilibrium point (x=-1, y=0) is a *stable center*, since the obtained values of the roots of the age equation for the linearization matrix at this point take the complex values λ_1=+i, λ_2=-i.

In accordance with the research results, a phase portrait of the system is constructed, where the obtained equilibrium points make it possible to determine the necessary boundaries of the construction of the phase portrait and its geometric features. The construction of the phase portrait is carried out by the method of numerical integration of the original system of equations using the Runge Kut method of the 4th order in the MATLAB system for engineering and mathematical calculations under different initial conditions, taking into account the prediction of the appearance of the phase portrait of the system based on the obtained equilibrium points. The set of phase trajectories obtained during a series of numerical integrations of the system visualizes the behavior of the system predicted above. The results of the study are presented in Fig. 1.

Let's analyze them. Figure 1 shows the equilibrium points and the direction of movement along the phase trajectories. It is thus clearly visible that orbital motion is observed in the vicinity of the stable center x=-1, y=0 In the vicinity of the saddle point x=0, y=0 (Fig. 1) clearly expressed directions of attraction and repulsion to the unstable saddle point are visible. The nature of the phase trajectories at a certain distance from the determined equilibrium points is determined by the general influence of the attraction and repulsion directions and the orbital motion caused by the presence of both equilibrium points.

Comparison of the results obtained by different methods, namely, the phase portrait constructed by numerical integration and the graph of the gradient field of the system gives a complete coincidence, which confirms the reliability of the obtained results. That is, conducting research using different methods makes it possible to obtain reliable results and more fully present the behavior of the system.

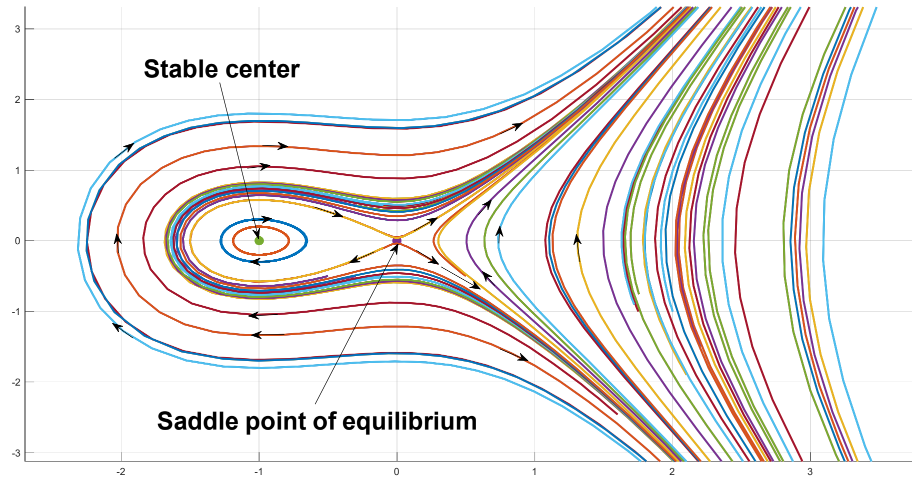

Fig. 1 The phase portrait of the given system is obtained by numerical integration at different values of the initial conditions

To identify the direction of flow, we will plot the graphs of the gradient field.

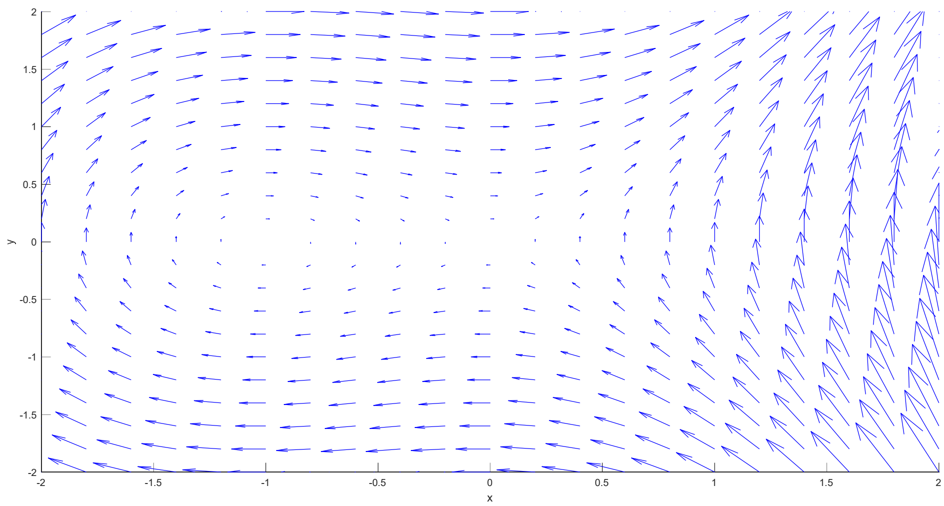

Fig. 2 Graph of the gradient field of the given system

Conclusions. As a result of the tasks from the discipline "Software for mathematical calculations", students master a wide range of methods of researching mathematical models of various systems. One of the examples of using the knowledge gained by students in the process of mastering the course is the application of a geometric interpretation of the behavior of the system with the construction of its phase portrait, which is demonstrated in this study of the given system. The combination of mathematical apparatus and modern software gives good results and is, in most cases, the only possible means of conducting comprehensive research of mathematical models of complex systems. Using the MATLAB system for such mathematical and engineering calculations and mastering the tools of this package gives the student the opportunity to acquire professional competencies that meet the requirements of modern production. The acquired knowledge and acquired skills will be used by students in further studies, and will later help in independent scientific activity.

References

1. Бондаренко З. В.: Курс вищої математики з комп'ютерною підтримкою. Диференціальні рівняння : Навчальний посібник (З. В. Бондаренко, В. І. Клочко). - МОН України. - Вінниця: ВНТУ, 2004. - 130 с.
2. Лазарєв Ю. Ф. Моделювання на ЕОМ. Навч. посібник. – К.: Корнійчук, 2007.-290 с.
3. Лазарєв Ю. Ф. Початки програмування у середовищі MatLAB: Навч. посібник. – К.:"Корнійчук", 1999. – 160 с.
4. Краснопрошина А. А., Репникова Н. Б., Ильченко А. А. Современный анализ систем управления с применением MATLAB, Simulink, Control System: Учебное пособие. - К.: "Корнійчук", 1999. - 144 с.
5. Клочко, В. І.: Вища математика. Функції багатьох мінних. Диференціальні рівняння. Тестові завдання: навчальний посібник (В. І. Клочко, З. В.Бондаренко, С. А. Кирилащук). - Вінниця : ВНТУ, 2012. - 84 с.
6. Овчинников П.П. та ін. Вища математика: Підручник. У 2 ч. Ч.1: Лінійна і векторна алгебра: Аналітична геометрія: Вступ до математичного аналізу: Диференціальне і інтегральне числення (Овчинников П.П., Яремчук Ф.П., МихайленкоВ.М.; За заг. ред. Овчинникова П.П.)- К.: Техніка, 2003. - 600 с.
7. Овчинников П.П., Михайленко В.М. Вища математика: Підручник. У 2 ч. Ч.2: Диференціальні рівняння. Операційне числення. Ряди та їх застосування. Стійкість за Ляпуновим. Рівняння математичної фізики. Оптимізація і керування. Терія ймовірностей. Числові методи; За заг. ред. Овчинникова П.П. - К.: Техніка, 2004. -792 с.
8. Хусаїнов Д. Я., Шатирко А. В. Основи нелінійної динаміки: Посібник для студентів спеціальності "Прикладна математика" К.: Видавничо-поліграфічний центр "Київський університет", 2017. – 159 с.
9. Гнітецька Т., Гнітецька Г, Бондаренко Ю, Губар О Дослідження геометричної структури фазових потоків динамічної системи з хаотичною поведінкою . Прикладна геометрія та інженерна графіка, No105, 2024,С. 81 - 93.