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CONSTRUCTION OF SPATIAL PYTHAGOREAN HODOGRAPH CURVES BY GAUSS-RADAU POLYGONS

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Abstract. *The method for constructing spatial polynomial Pythagorean hodograph curves (PH curves) from the given Gauss-Radau polygon is considered. An equation has been found that describes infinitely many RH curves that have the same Gauss-Radau polygon.*

Key words: *Pythagorean hodograph curves; Gauss-Radau quadrature; Gauss-Radau polygon; interpolation.*

Problem statement. Spatial Pythagorean-hodograph (PH) curves were first studied independently by Farouki and Dietz [1], and by Dietz et al. [2]. They established the conditions under which the arc length derivative of such curves with respect to the curve parameter is a polynomial rather than a radical function. This property offers numerous computational advantages in areas such as computer-aided geometric design, animation, robotics, and motion planning. The polynomial speed functions of PH curves allow for exact arc length computation, rational offset generation, and the determination of rational unit tangent vectors. PH curves are effective for interpolation and approximation of space curves, especially when a specific shape is required. The problem of constructing spatial PH curves with a given shape is of important practical importance.

Analysis of recent research. To control the shape of PH curves, the use of a straightening control polygon based on Gauss-Lobatto quadrature was proposed in [3]. This approach preserves the desirable properties of PH curves while avoiding the drawbacks associated with modifying Bézier control points. As an alternative to the Bézier polygon, the Gauss-Legendre polygon of a PH curve is considered in [4], [5], with its vertices obtained by evaluating derivatives at the nodes of the Gauss-Legendre quadrature. A drawback of the Gauss-Legendre polygon is that it does not define tangent vectors at the endpoints. PH curves with the same Gauss-Legendre polygon may have different endpoint tangents, since all quadrature nodes are interior points.

Formulation of goals. In this work, we consider the construction of spatial PH curves using a control polygon derived from Gauss-Radau quadrature, building on the methods described in [4], [5]. Employing the Gauss-Radau polygon, as an alternative to Gauss-Legendre and Gauss-Lobatto polygons, enables the development of another class of adaptive-shape spatial PH curves. A notable advantage of this method is that the Gauss-Radau polygon naturally defines the initial tangent vector of the PH curve, since the initial point of the Gauss-Radau quadrature is a predetermined node.

Main part. Let a spatial polynomial PH curve be defined by a quaternion-valued function $p(t) = x(t)i + y(t)j + z(t)k$, $x(t), y(t), z(t) \in \mathbf{R}[t]$, $t \in \mathbf{R}$. We consider the concept of a Gauss-Radau polygon based on the Gauss-Radau quadrature over the interval $[0;1]$. The Gauss-Radau quadrature with n nodes for an integrable function $f(x)$ defined on the interval $[0;1]$, is a quadrature formula given by the expression

$$\int_0^1 f(x)dx = \omega_0 f(0) + \sum_{i=1}^{n-1} \omega_i f(x_i) + \varepsilon_n .$$

Nodes x_i for $i = 1,...,n-1$ are the roots of the polynomial $\dfrac{P_{n-1}(x) + P_n(x)}{x+1}$, where $P_k(x)$ is the k-th Legendre polynomial of degree k. The weighting coefficients have the form:

$$\omega_0 = \frac{1}{n^2}, \quad \omega_i = \frac{1 - x_i}{2\left(nP_{n-1}(x_i)\right)^2} = \frac{1}{2(1-x_i)\left(P'_{n-1}(x_i)\right)^2}, \ (x_i \neq 1), i = 1,...,n-1 .$$

Residual term is $\varepsilon_n = \dfrac{2^{2n-2} n\left((n-1)!\right)^4}{\left((2n-1)!\right)^3} f^{(2n-1)}(\xi), \ (0 < \xi < 1)$.

The Gauss-Radau polygon of a curve $p(t)$ with m edges is defined as $G_m(p) = [r_0, r_1, ..., r_m]$, where

$$\begin{aligned} &r_0 = p(0), \\ &r_{k+1} = r_k + \omega_k p'(x_k), \text{ for } k = 0,...,m-2 . \\ &r_m = r_{m-1} + \omega_{m-1} p'(x_{m-1}) + \varepsilon_m . \end{aligned} \qquad (1)$$

In [6] it is shown that the derivative $p'(t)$ is the $*$-square of some quaternion polynomial $A(t)$ of degree n: $p'(t) = A(t)^{2*} = A(t) i \overline{A(t)}$. We write the polynomial $A(t)$ in the form of a Bezier polynomial $A(t) = \sum_{l=0}^{n} B_l^n(t) A_l$, where $B_l^n(t) = C_n^l (1-t)^{n-l} t^l$ are Bernstein polynomials.

Let $[r_0, r_1, ..., r_{n+1}]$ be the Gauss-Radau polygon of the curve $p(t)$ with $n+1$ edges. Let us write equation (1):

$$\Delta r_k = \omega_k p'(x_k) = \omega_k \left(\sum_{l=0}^{n} B_l^n(x_k) A_l \right)^{2*}, \ k = 0,...,n-1,$$

$$\Delta r_n = \omega_n p'(x_n) + E_n = \omega_n \left(\sum_{l=0}^{n} B_l^n(x_n) A_l \right)^{2*} + \varepsilon_n .$$

Each of these equations can be represented in the form ([5]):

$$\sum_{l=0}^{n} B_l^n(x_k)A_l = \sqrt[*]{\left(\frac{\Delta r_k}{\omega_k}\right)}\, e^{i\varphi_k}, \; k=0,\ldots,n-1, \tag{2}$$

$$\sum_{l=0}^{n} B_l^n(x_n)A_l = \sqrt[*]{\left(\frac{\Delta r_n - \varepsilon_n}{\omega_n}\right)}\, e^{i\varphi_n}, \text{ where } \varphi_0,\varphi_1,\ldots,\varphi_n \in \mathbf{R}.$$

Equations (2) form a linear system with a unique solution $A_0, A_1,\ldots,A_n$, so the quaternion polynomial $A(t) = A[\varphi_0,\ldots,\varphi_n](t)$ depends on the $n+1$ a set of free parameters $\varphi_0,\varphi_1,\ldots,\varphi_n \in \mathbf{R}$. We show that the function $A[\varphi_0,\ldots,\varphi_n](t)$ with a fixed parameter $\varphi_0 = 0$ describes all possible spatial PH curves in which the Gauss-Radau polygon with $n+1$ edges is the given polygon $[r_0, r_1,\ldots,r_{n+1}]$. Thus, the polynomial $p[0,\varphi_1,\ldots,\varphi_n](t)$, as a solution to the equation $p'(t) = A(t)^{2*}$, specifies all spatial PH curves of degree $2n+1$ such that $G_{n+1}(p) = [r_0, r_1,\ldots,r_{n+1}]$.

Conclusions. Constructing spatial polynomial PH curves from a Gauss-Radau polygon yields infinitely many curves with the same polygon, defining the tangent only at the initial point. This is useful for spline interpolation, where the initial tangent at internal nodes must be specified.

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