MATHEMATICAL APPROACH TO THE PROBLEM OF CONSTRUCTING GEOMETRIC CONJUGATIONS

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***Abstract** — this paper presents a mathematical approach to the construction of various types of conjugations between circular arcs and straight lines, which can be used to expand the capabilities of algorithms for constructing conjugates using CAD systems.*

***Keywords** — conjugation, circle, straight line, arc, conjugation points, mathematical approach.*

Problem statement. In modern mathematical modeling and applied geometry, the problem of constructing geometric conjugation plays an important role. Such constructions are widely used in engineering graphics, computer modeling, architectural design, and mechanical engineering.

Despite the development of numerous methods for constructing geometric conjugation, there remains a need for in-depth analysis of this process from a mathematical perspective. In particular, relevant issues include the formalization of transition concepts, the development of algorithms for various cases (such as transitions between an arc and a line, or between two arcs), and the study of the conditions for existence and uniqueness of such constructions.

Thus, the problem considered in this study is the formalization and mathematical justification of the processes for constructing conjugation between various geometric objects. This will improve the accuracy and efficiency of applying these methods in practical tasks.

Analysis of Recent Studies. Current research is focused on enhancing the capabilities of transition construction algorithms within the framework of computer-aided design (CAD) systems. These studies highlight the importance of not only geometric accuracy, but also numerical stability and computational efficiency.

Formulation of goals (Problem statement). The goal of this publication is to develop an understanding of the computational processes involved in the construction of conjugation.

The main part. The surfaces of many parts on drawings are depicted with lines that smoothly transition from one to another. Smooth transitions are determined by the structural properties of the parts, their manufacturing technology, functional purposes, etc.

The smooth transition of a straight line or arc into another is called conjugation. To construct a conjugation using an arc, you need to find the center of conjugation and conjugation points. Let's analyze some cases of conjugation of arcs and straight lines.

The construction of conjugations is based on the following theoretical provisions:

1. A line is tangent to a circle if it is perpendicular to the radius drawn to the point of contact (Fig. 1, a). To draw a tangent line from a given point *A* to a circle with center at point *O*, construct a right angle *OKA* (Fig. 1, b) as the interior angle of an auxiliary circle of diameter *OA*.

2. Two circles will be tangent if the point of contact *K* is on the line connecting (Fig. 1, c) the centers O_1 i O_2. The contact of a circle can be external (Fig. 1, c) and internal (Fig. 1, d).

3. The center of the arc of conjugation *O* of two circles of the same radius lies on the median perpendicular to the line connecting their centers O_1 i O_2. The conjugation of two circles by an arc can be external or internal.

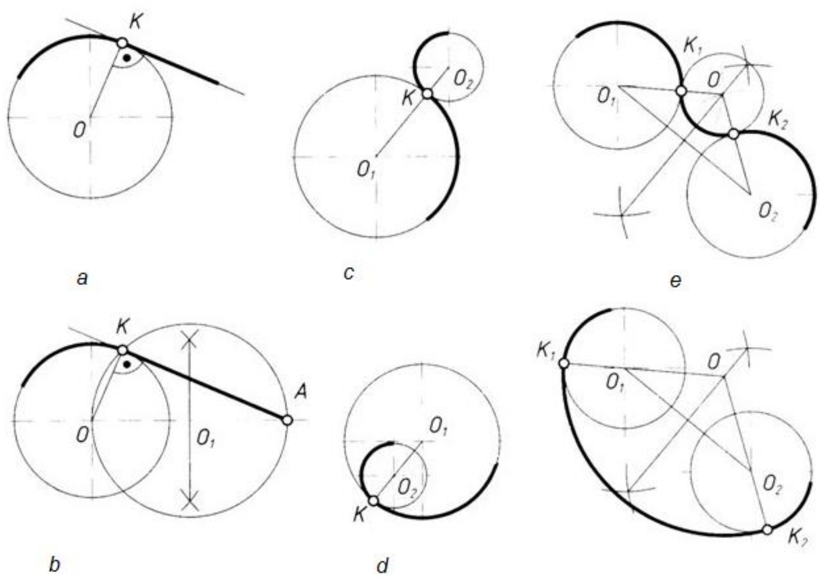

Fig. 1. Construction of conjugation

Let us consider the internal conjugation of two circles from the point of view of geometric constructions. From the centers O_1 and O_2 auxiliary arcs are drawn with a compass spread equal to the sum of the radii for the given conjugation arc (Fig. 2, a). The radius of the arc drawn from the center O_1 is equal to $R_1 + R_3$. The conjugation center is located at the intersection of the auxiliary arcs – point O_3.

By connecting point O_1 with point O_3 and point O_2 with point O_3, with straight lines we find the conjugation points *A* and *B* (fig. 2, *b*).

From point O_3 we build an arc between points A and B with a compass deviation equal to R_3.

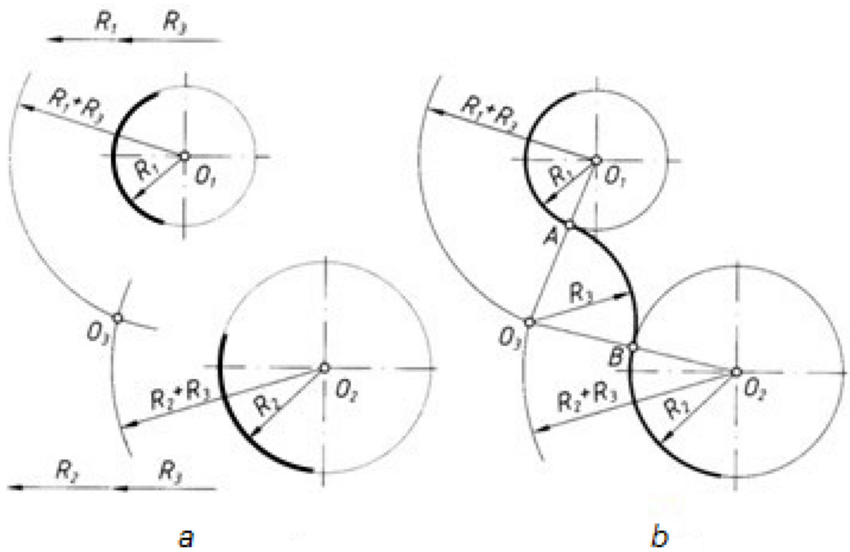

Fig. 2. Construction of the inner conjugate of two circles

Let's analyze the internal conjugation of two circles from a mathematical point of view. Given two circles on the coordinate plane with radii R_1 and R_2, with the coordinates of the centers $O_1(x_1;y_1)$ and $O_2(x_2;y_2)$ respectively, and a tangent circle with a known radius R_3 and unknown center O_3 with coordinates $(x_3;y_3)$ (Fig. 3). Now we need to find the coordinates $(x_3;y_3)$. Consider circles 1 and 3. Since they are tangent, the distance between their centers must be equal to the sum of their radii. Thus, we get:

$$O_1O_3 = R_1 + R_3 \qquad (1)$$

On the other side, the length of the segment O_1O_3 can be found as the distance between two points [4, p.77]:

$$(O_1O_3)^2 = (x_3 - x_1)^2 + (y_3 - y_1)^2 \qquad (2)$$

Substituting equation (1), we get:

$$(R_1 + R_3)^2 = (x_3 - x_1)^2 + (y_3 - y_1)^2 \qquad (3)$$

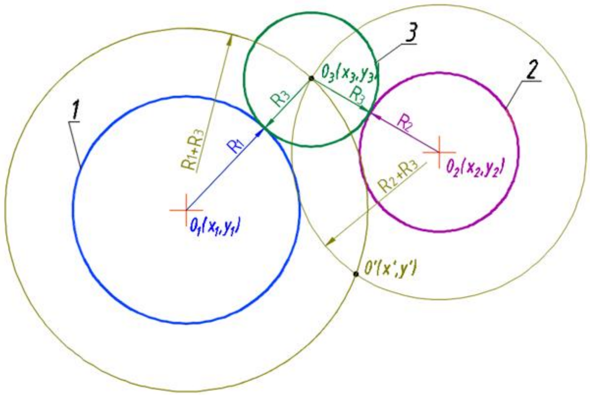

Fig. 3. Geometric interpretation of the mathematical description of the conjugation of two circles

Based on similar considerations, we obtain equations for circles 2 and 3:

$$(R_2 + R_3)^2 = (x_3 - x_2)^2 + (y_3 - y_2)^2 \quad (4)$$

Since the numbers x_3 and y_3 must satisfy equations (2) and (3) at the same time, it is possible to make a system of two equations:

$$\begin{cases} (R_1 + R_3)^2 = (x_3 - x_1)^2 + (y_3 - y_1)^2 \\ (R_2 + R_3)^2 = (x_3 - x_2)^2 + (y_3 - y_2)^2 \end{cases} \quad (5)$$

As a result of solving the system, we will obtain two pairs of possible real solutions. One pair of coordinates will correspond to the point O_3, and the other to – O_3*'*. he number of solutions is confirmed graphically.

After finding the coordinates of the center of the third circle, it remains to find the coordinates of the conjugation points (in this work, we will consider finding the coordinates of the conjugation points only for one of the two cases of the position of the third circle (O_3), in the case of O_3*'* the coordinates of the conjugation points are found similarly). Let's mark the points of conjugation $A(x_A; y_A)$ and $B(x_B; y_B)$ (Fig. 4).

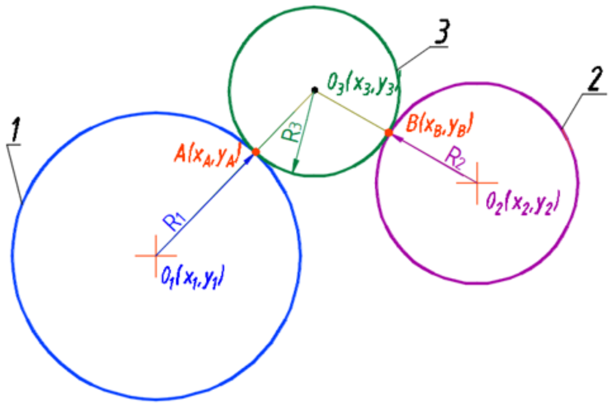

Fig. 4. Geometric interpretation of the definition of conjugation points

Consider point A. The points of conjugation correspond to the points of contact of two circles, so point A will belong to circle 1 and circle 3 at the same time. So the distance from A to O_1 should be equal to R_1, and from A to O_3 – R_3. Let's use the formula for finding the distance between two points [4, p. 77] and create a system:

$$\begin{cases} R_1^2 = (x_A - x_1)^2 + (y_A - y_1)^2 \\ R_3^2 = (x_A - x_3)^2 + (y_A - y_3)^2 \end{cases} \quad (6)$$

After solving the system, we will obtain only one pair of real roots, this is confirmed graphically (circles 1 and 3 have only 1 point of contact).

Based on similar considerations, let's make a system for point:

$$\begin{cases} R_2^2 = (x_B - x_2)^2 + (y_B - y_2)^2 \\ R_3^2 = (x_B - x_3)^2 + (y_B - y_3)^2 \end{cases} \quad (7)$$

Conclusions. The article provides a mathematical justification of the process of creating a conjugation, which can be the basis of CAD algorithms, where not only geometric correctness is important, but also numerical stability and efficiency of calculations.

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